Extending Proof Tree Preserving Interpolation to Sequences and Trees (Work in Progress)

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Extending
Proof Tree Preserving Interpolation
to
Proof Tree Preserving Tree Interpolation

Outline

Uses of Tree Interpolation

$$
\begin{tikzcd}
n = 0 \arrow[d] \\
n \leq 0 \arrow[d] \\
res = 0 \arrow[d] \\
res < n
\end{tikzcd}
$$

- Hoare-style program verification [Henzinger 04]

procedure $f(n)$ returns res
if $(n <= 0)$
    $res := 0$

assert $res >= n$

Uses of Tree Interpolation

$$
\begin{array}{c}
n = 0 \\
\downarrow \top \\
n \leq 0 \\
\downarrow \top \\
res = 0 \\
\downarrow n \leq res \\
res < n
\end{array}
$$

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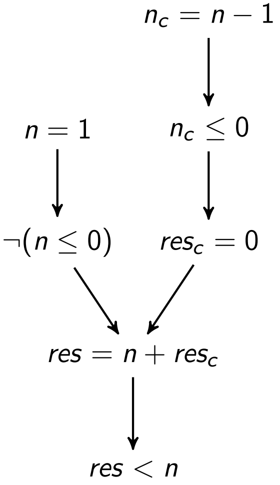
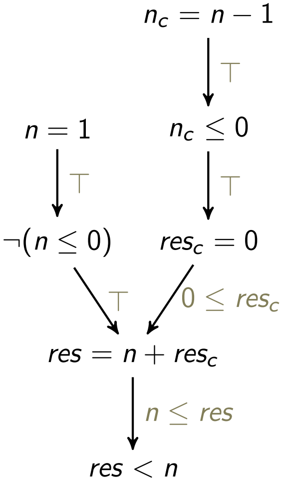
Uses of Tree Interpolation



- Hoare-style program verification [Henzinger 04, Heizmann 10]

procedure $f(n)$ returns res
if $(n <= 0)$
    $res := 0$
else
    $res := n + \textbf{call } f(n-1)$
assert $res >= n$



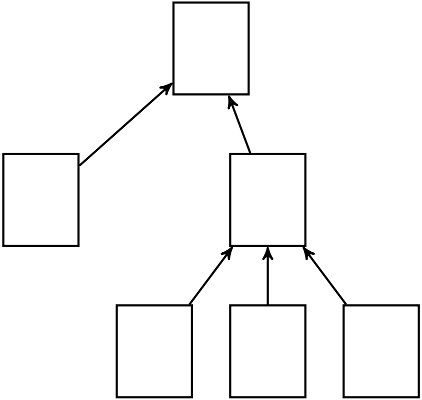
- Hoare-style program verification [Henzinger 04, Heizmann 10]

procedure $f(n)$ returns res
if $(n <= 0)$
  $res := 0$
else
  $res := n +$ **call** $f(n-1)$
assert $res >= n$

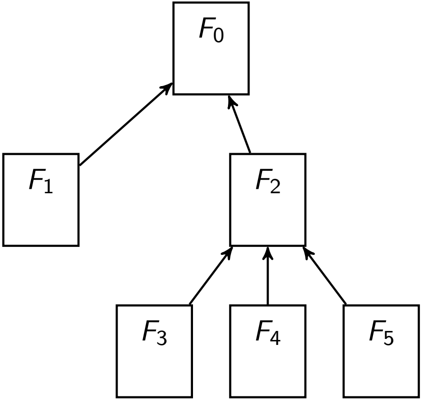
Uses of Tree Interpolation

- Hoare-style program verification [Henzinger 04, Heizmann 10]
- Verification of multi-threaded programs and higher order programs [Rybalchenko 12]
- Incremental update checking [Sery 11]
- Solving non-recursive Horn clauses [Rybalchenko 11]
- Inductive Dataflow Graphs [Podelski 13]
- ...

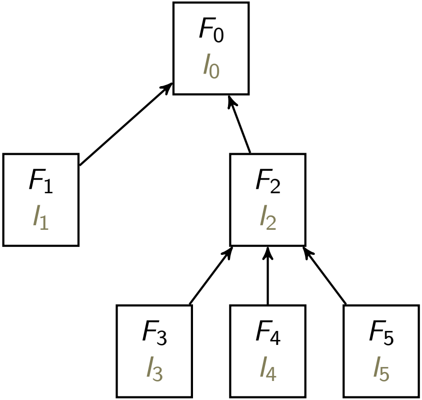
Tree Interpolation Problem



Tree Interpolation Problem



$\bigwedge F_i$ is unsatisfiable



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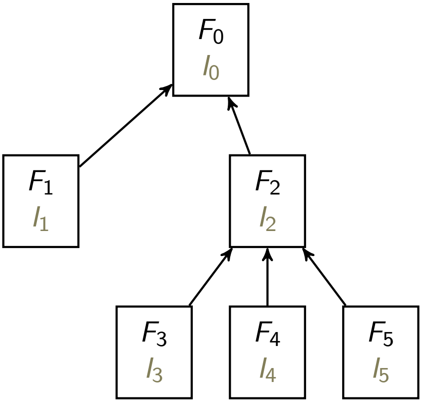

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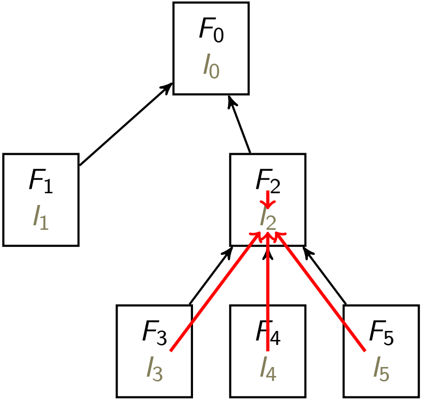
Tree Inductivity:

- $I_0 \equiv \bot$

Tree Interpolation Problem



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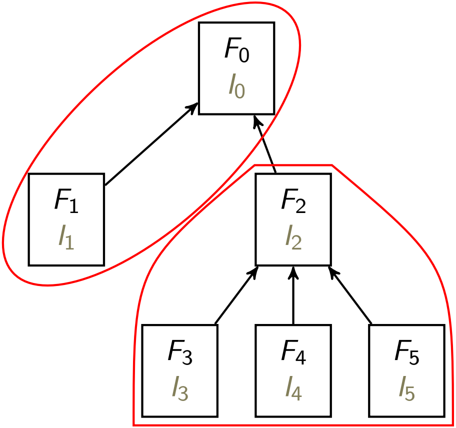
Tree Interpolation Problem



$\bigwedge F_i$ is unsatisfiable

Tree Inductivity:

- $I_0 \equiv \bot$
- Child interpolants and parent imply parent interpolant
- Interpolant only contains symbols occurring inside and outside the current subtree

Outline

For $A \wedge B \models_{\mathcal{T}} \bot$:

- $A \models_{\mathcal{T}} I$,
- $B \wedge I \models_{\mathcal{T}} \bot$,
- $symb(I) \subseteq symb(A) \cap symb(B)$

Binary Interpolation

For $A \land B \models_{\mathcal{T}} \bot$:

- $A \models_{\mathcal{T}} I$,
- $B \land I \models_{\mathcal{T}} \bot$,
- $symb(I) \subseteq symb(A) \cap symb(B)$

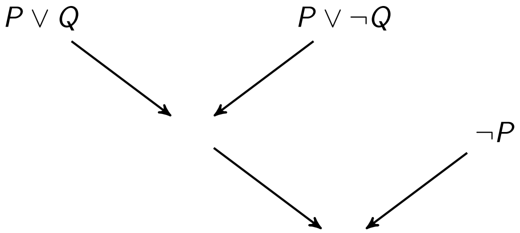
$$\begin{tikzcd} B \\ A \arrow[u] \end{tikzcd}$$

Outline

Resolution Refutation

Proof consists of

- leaves representing input clauses,



Proof consists of

- leaves representing input clauses,
- inner nodes derived by resolution

$$\frac{C_1 \vee \ell \qquad C_2 \vee \neg\ell}{C_1 \vee C_2}$$

$$\begin{tikzcd}
P \vee Q \arrow[dr] & & P \vee \neg Q \arrow[dl] & \\
& P \arrow[dr] & & \neg P \arrow[dl] \\
& & {} &
\end{tikzcd}$$

Proof consists of

- leaves representing input clauses,
- inner nodes derived by resolution

$$\frac{C_1 \lor \ell \qquad C_2 \lor \neg\ell}{C_1 \lor C_2}$$

- the root node representing the empty clause.

$$\begin{tikzcd}
P \lor Q \arrow[dr] & & P \lor \neg Q \arrow[dl] & \\
& P \arrow[dr] & & \neg P \arrow[dl] \\
& & \bot &
\end{tikzcd}$$

Labelled Resolution Refutation

Label each clause in the resolution refutation with *partial interpolant*

$$\begin{tikzcd}
P \lor Q \ : \ I_{P \lor Q} \arrow[dr] & & P \lor \neg Q \ : \ I_{P \lor \neg Q} \arrow[dl] & \\
& P \ : \ I_P \arrow[dr] & & \neg P \ : \ I_{\neg P} \arrow[dl] \\
& & \bot \ : \ I_\bot &
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& & \perp \ : \ I_{\perp} &
\end{tikzcd}
$$

- Syntactic rules for leaves

Labelled Resolution Refutation

Label each clause in the resolution refutation with *partial interpolant*

$$\ell \in A \; \frac{C_1 \vee \ell \ : \ I_1 \qquad C_2 \vee \neg\ell \ : \ I_2}{C_1 \vee C_2 \ : \ I_1 \vee I_2}$$

$$\ell \in B \; \frac{C_1 \vee \ell \ : \ I_1 \qquad C_2 \vee \neg\ell \ : \ I_2}{C_1 \vee C_2 \ : \ I_1 \wedge I_2}$$

$$\begin{tikzcd}
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- Syntactic rules for leaves
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Labelled Resolution Refutation

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$$\ell \in {\color{red}B}\ \frac{\begin{array}{c} C_1 \vee \ell \ : \ I_1 \\ C_2 \vee \neg\ell \ : \ I_2 \end{array}}{C_1 \vee C_2 \ : \ I_1 \wedge I_2}$$

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$I_\bot$ is desired interpolant.

Partial Interpolants

Partial interpolant I_C of clause C is interpolant of

$$A \wedge B \wedge \neg C$$

Partial interpolant I_C of clause C is interpolant of

$$A \land B \land \neg C$$

How to split $\neg C$?

Partial Interpolants

Partial interpolant I_C of clause C is interpolant of

$$A \wedge B \wedge \neg C$$

Define $\neg C \downharpoonleft A$ and $\neg C \downharpoonleft B$ such that

- $symb(\neg C \downharpoonleft A) \subseteq symb(A)$
- $symb(\neg C \downharpoonleft B) \subseteq symb(B)$
- $\neg C \leftrightarrow \neg C \downharpoonleft A \wedge \neg C \downharpoonleft B$

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- $\neg C \leftrightarrow \neg C \downharpoonleft A \land \neg C \downharpoonleft B$

Partial interpolant I_C is interpolant of
$A \land ((\neg C) \downharpoonleft A)$ and $B \land ((\neg C) \downharpoonleft B)$.

Outline

Additional Leaves

- Theory lemmas
- Theory combination lemmas

$$x \leq y \vee x \neq y$$
$$x \geq y \vee x \neq y$$
$$x < y \vee x > y \vee x = y$$

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- Theory combination lemmas

$$x \leq y \vee x \neq y$$
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might contain literals that are not in the input formulas

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  - theory combination (Nelson-Oppen, Ackermannization),
  - cuts and extended branches used to solve integer arithmetic,
  - ...

Mixed Literals

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  - theory combination (Nelson-Oppen, Ackermannization),
  - cuts and extended branches used to solve integer arithmetic,
  - ...

What is $a = b \downharpoonleft A$ and $a = b \downharpoonleft B$?

Interpolation and Mixed Literals

Purification:
replace $a \leq b$ by $a \leq x \land x \leq b$
similar to purification in Nelson-Oppen

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replace $a \leq b$ by $a \leq x \wedge x \leq b$
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Interpolation:
Remove purification variable on resolution:

$$\frac{C_1 \vee a \leq b \;:\; I_1(x_1) \qquad C_2 \vee \neg(a \leq b) \;:\; I_2(x_2)}{C_1 \vee C_2 \;:\; I_3}$$

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Rules for uninterpreted functions and linear arithmetic [TACAS 2013]

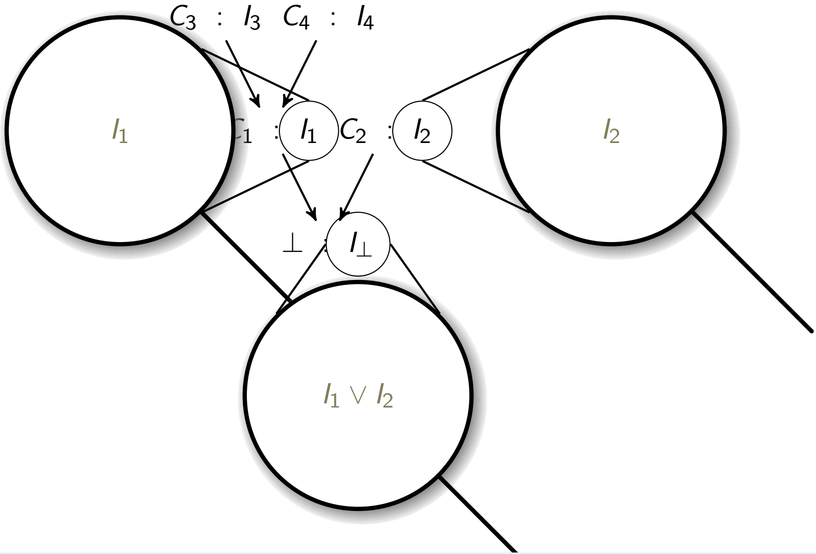
Outline

Idea

Binary Interpolation:

$$
\begin{tikzcd}
C_3 : I_3 \arrow[dr] & & C_4 : I_4 \arrow[dl] & \\
& C_1 : I_1 \arrow[dr] & & C_2 : I_2 \arrow[dl] \\
& & \bot : I_\bot &
\end{tikzcd}
$$

Binary Interpolation:



Idea

Tree Interpolation:

$$
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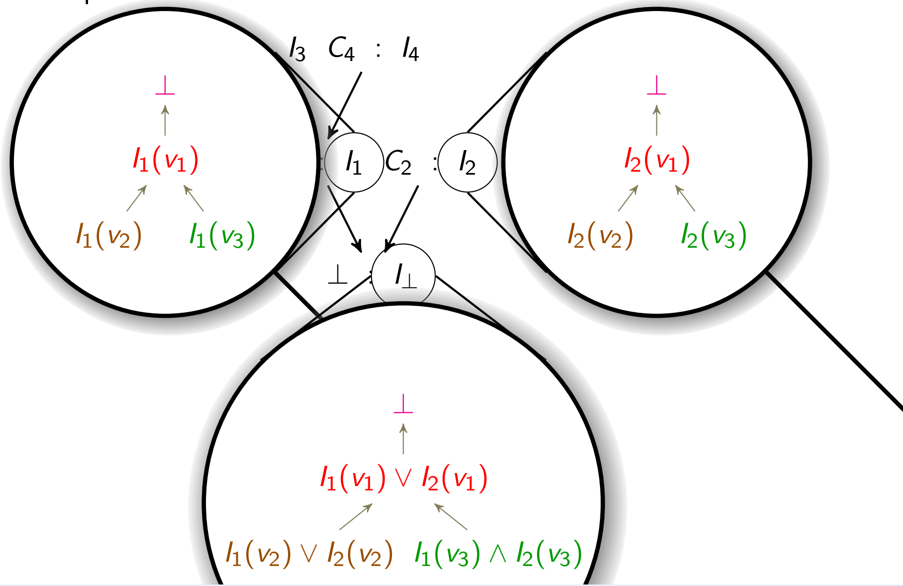

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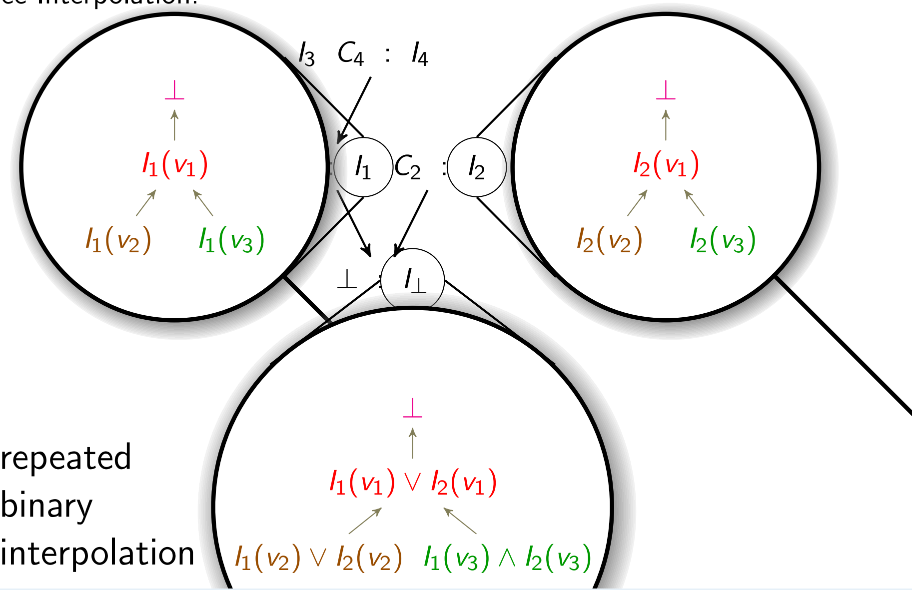


repeated binary interpolation

Partial tree interpolant I_C for clause C is tree interpolant of

$$\begin{tikzcd} & F_0 & & \\ & F_1 \arrow[u] & & \land \neg C \\ F_2 \arrow[ur] & & F_3 \arrow[ul] & \end{tikzcd}$$

How to split $\neg C$?

Partial Tree Interpolants

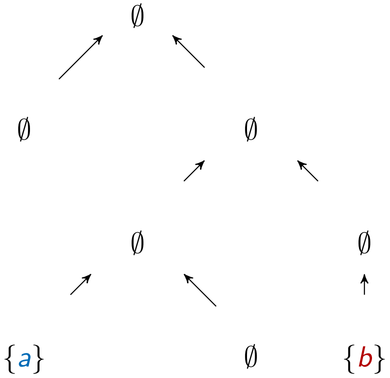
Partial tree interpolant I_C for clause C is tree interpolant of

$$\begin{tikzcd}[column sep=small]
 & F_0 \land ((\neg C) \downharpoonleft v_0) & \\
 & F_1 \land ((\neg C) \downharpoonleft v_1) \arrow[u] & \\
F_2 \land ((\neg C) \downharpoonleft v_2) \arrow[ur] & & F_3 \land ((\neg C) \downharpoonleft v_3) \arrow[ul]
\end{tikzcd}$$

- One purification function per node
- $\ell \leftrightarrow \exists \overline{x}.\ \bigwedge_v \ell \downharpoonleft v$

Projection of Mixed Literals

- one auxiliary variable for every node in which literal is mixed
- projection of $a = b$:



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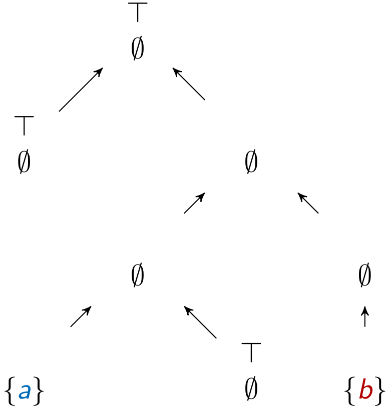

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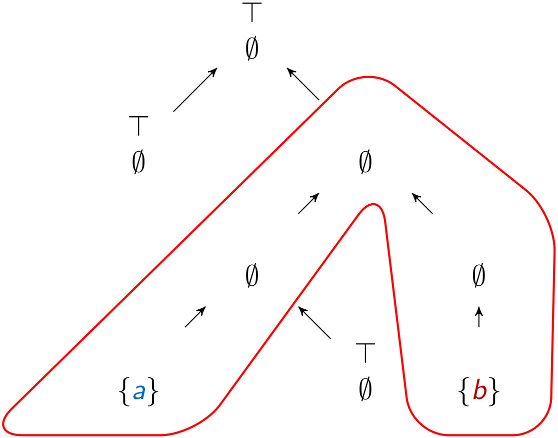

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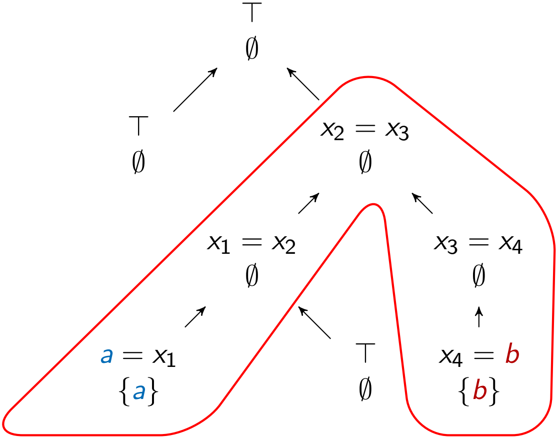


Outline

$$
\begin{tikzcd}
& \begin{array}{c}\{q, r\}\\ q \neq r\end{array} & \\
\begin{array}{c}\{c, d\}\\ c = d\end{array} \arrow[ur] & & \begin{array}{c}\{b, d, r, f(\cdot)\}\\ d = b \wedge f(b) = r\end{array} \arrow[ul] \\
\begin{array}{c}\{a, c, q, f(\cdot)\}\\ a = c \wedge q = f(a)\end{array} \arrow[u] & &
\end{tikzcd}
$$

$$
\frac{a = b \vee a \neq c \vee c \neq d \vee d \neq b \qquad a \neq b \vee q \neq f(a) \vee f(b) \neq r \vee q = r}{a \neq c \vee c \neq d \vee d \neq b \vee q \neq f(a) \vee f(b) \neq r \vee q = r}
$$

Interpolation Problem and Proof Excerpt

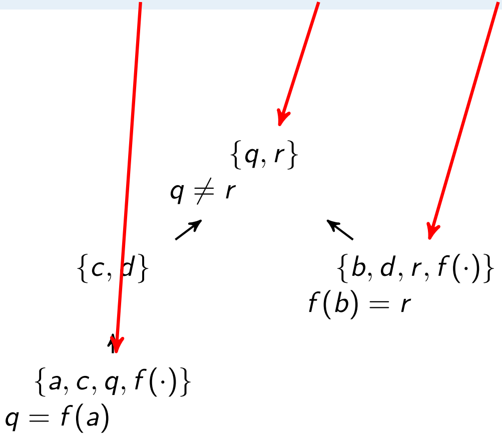
$$\begin{tikzcd}[column sep=small]
& \begin{matrix}\{q, r\} \\ q \neq r\end{matrix} & \\
\begin{matrix}\{c, d\} \\ c = d\end{matrix} \arrow[ur] & & \begin{matrix}\{b, d, r, f(\cdot)\} \\ d = b \wedge f(b) = r\end{matrix} \arrow[ul] \\
\begin{matrix}\{a, c, q, f(\cdot)\} \\ a = c \wedge q = f(a)\end{matrix} \arrow[u] & &
\end{tikzcd}$$

$$\frac{a = b \vee a \neq c \vee c \neq d \vee d \neq b \qquad a \neq b \vee q \neq f(a) \vee f(b) \neq r \vee q = r}{a \neq c \vee c \neq d \vee d \neq b \vee q \neq f(a) \vee f(b) \neq r \vee q = r}$$

Projection: $a = b \wedge q = f(a) \wedge q \neq r \wedge f(b) = r$

$$
\begin{tikzcd}
 & \{q, r\} & \\
\{c, d\} \arrow[ur] & & \{b, d, r, f(\cdot)\} \arrow[ul] \\
\{a, c, q, f(\cdot)\} \arrow[u] & &
\end{tikzcd}
$$

Projection: $a = b \wedge q = f(a) \wedge q \neq r \wedge f(b) = r$



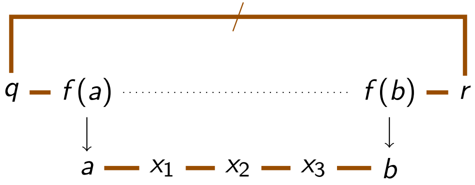
$$\begin{array}{ccc}
 & \{q, r\} & \\
 & q \neq r \wedge x_2 = x_3 & \\
\nearrow & & \nwarrow \\
\{c, d\} & & \{b, d, r, f(\cdot)\} \\
x_1 = x_2 & & f(b) = r \wedge x_3 = b \\
\uparrow & & \\
\{a, c, q, f(\cdot)\} & & \\
q = f(a) \wedge a = x_1 & &
\end{array}$$

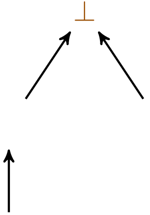
Interpolation: $a = b \land q = f(a) \land f(b) = r \land q \neq r$

$q \neq r \land x_2 = x_3$

$x_1 = x_2$ $\quad$ $f(b) = r \land x_3 = b$

$q = f(a) \land a = x_1$

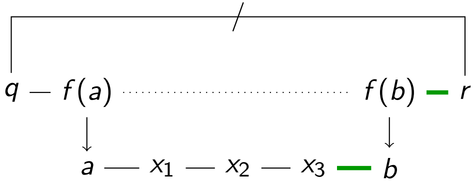


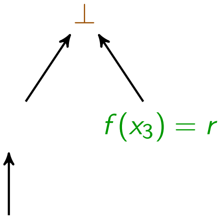


$q \neq r \wedge x_2 = x_3$

$x_1 = x_2$ $\quad$ $f(b) = r \wedge x_3 = b$

$q = f(a) \wedge a = x_1$

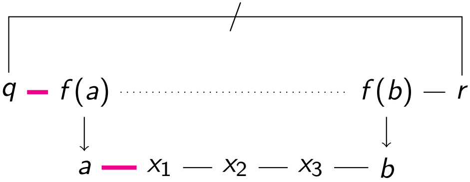




Interpolation: $a = b \wedge q = f(a) \wedge f(b) = r \wedge q \neq r$

$$\begin{tikzcd}
& q \neq r \wedge x_2 = x_3 & \\
x_1 = x_2 \arrow[ur] & & f(b) = r \wedge x_3 = b \arrow[ul] \\
\color{magenta}{q = f(a) \wedge a = x_1} \arrow[u] & &
\end{tikzcd}$$

$$\begin{tikzcd}
& \bot & \\
{} \arrow[ur] & & \color{green}{f(x_3) = r} \arrow[ul] \\
\color{magenta}{q = f(x_1)} \arrow[u] & &
\end{tikzcd}$$



$$q \neq r \wedge x_2 = x_3$$

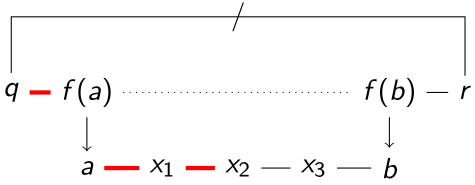
$$x_1 = x_2 \qquad f(b) = r \wedge x_3 = b$$

$$q = f(a) \wedge a = x_1$$

$$\perp$$

$$q = f(x_2) \qquad f(x_3) = r$$

$$q = f(x_1)$$



Projection: $a = c \land c = d \land d = b \land a \neq b$

$$
\begin{tikzcd}
 & \{q, r\} & \\
\{c, d\} \arrow[ur] & & \{b, d, r, f(\cdot)\} \arrow[ul] \\
\{a, c, q, f(\cdot)\} \arrow[u] & &
\end{tikzcd}
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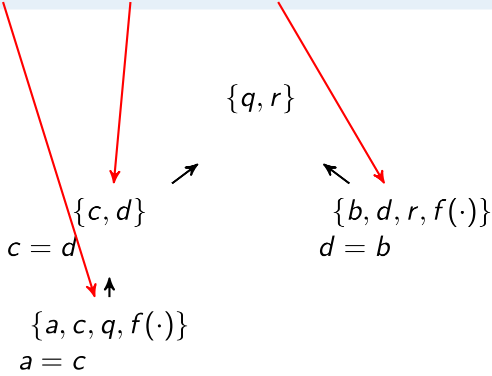


Projection: $a = c \wedge c = d \wedge d = b \wedge a \neq b$

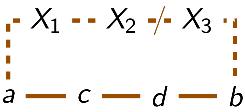
$$
\begin{tikzcd}
 & \begin{array}{c} \{q, r\} \\ X_2 \cap X_3 = \emptyset \end{array} & \\
\begin{array}{c} \{c, d\} \\ c = d \wedge X_1 \subseteq X_2 \end{array} \arrow[ur] & & \begin{array}{c} \{b, d, r, f(\cdot)\} \\ d = b \wedge b \in X_3 \end{array} \arrow[ul] \\
\begin{array}{c} \{a, c, q, f(\cdot)\} \\ a = c \wedge a \in X_1 \end{array} \arrow[u] & &
\end{tikzcd}
$$

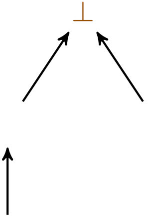
- X_1, X_2, X_3 set-valued
- X_i separates a and b
- No reasoning about sets required in the solver

$$X_2 \cap X_3 = \emptyset$$

$$c = d \wedge X_1 \subseteq X_2 \qquad d = b \wedge b \in X_3$$

$$a = c \wedge a \in X_1$$





$$X_2 \cap X_3 = \emptyset$$

$$c = d \land X_1 \subseteq X_2 \qquad d = b \land b \in X_3$$

$$a = c \land a \in X_1$$

$$X_1 \; \text{--} \; X_2 \; \not\text{--} \; X_3$$

$$a \;\text{—}\; c \;\text{—}\; d \;\text{—}\; b$$

$$\perp$$

$$d \in X_3$$

Interpolation: $a = c \wedge c = d \wedge d = b \wedge a \neq b$

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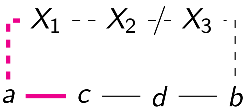

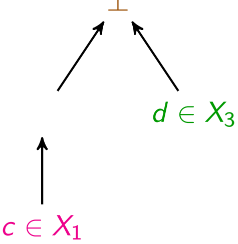

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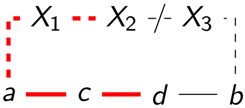

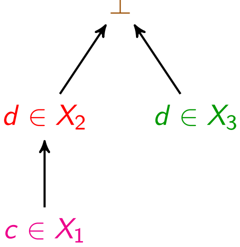

Magic Rule for Resolution on Mixed Equalities

- partial interpolant for $C_1 \vee a = b$ has form $I_1[s \in X]$
  "If $s \in X$ holds, then $s = a$ resp. $s = b$ (whichever is in the subtree)"
- partial interpolant for $C_2 \vee a \neq b$ has form $I_2(x)$
  "$I_2(x)$ holds for a resp. b (whichever is in the subtree)"

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- partial interpolant for the resolvent $C_1 \vee C_2$

$$I_1[I_2(s)]$$

Interpolating the Resolution Step

$$\bot \qquad\qquad \bot$$

$$C_1 \lor a = b \;:\; d \in X_2 \;\; d \in X_3 \qquad\qquad C_2 \lor a \neq b \;:\; q = f(x_2) \;\; f(x_3) = r$$

$$c \in X_1 \qquad\qquad q = f(x_1)$$

$$C_1 \lor C_2 \;:$$

Interpolating the Resolution Step

$$\begin{tikzcd}[column sep=small]
 & & \bot & \\
C_1 \lor a = b \; : & d \in X_2 \arrow[ur] & & d \in X_3 \arrow[ul] \\
 & c \in X_1 \arrow[u] & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & \bot & \\
C_2 \lor a \neq b \; : & q = f(x_2) \arrow[ur] & & f(x_3) = r \arrow[ul] \\
 & q = f(x_1) \arrow[u] & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & \bot & \\
C_1 \lor C_2 \; : & q = f(d) \arrow[ur] & & f(d) = r \arrow[ul] \\
 & q = f(c) \arrow[u] & &
\end{tikzcd}$$

Outline

Conclusion

- We extended our interpolation scheme to sequence and tree interpolation.
- Tree interpolation is repeated binary interpolation.
- Scheme computes quantifier-free interpolants in the combination of UF and LA, in particular in QF_UFLIA.
- No need to manipulate resolution proof.
- Independent of the solver or proof search.
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- Correctness proofs still work in progress.
- Scheme is implemented in SMTInterpol.

`http://ultimate.informatik.uni-freiburg.de/smtinterpol`

Thanks for your attention